Planar Rook Algebras and Tensor Representations of $\mathfrak{gl}(1|1)$

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Abstract

We establish a connection between planar rook algebras and tensor representations $\mathsf{V}^{\otimes k}$ of the natural two-dimensional representation V of the general linear Lie superalgebra $\mathfrak{gl}(1|1)$. In particular, we show that the centralizer algebra $\mathsf{End}_{\mathfrak{gl}(1|1)}(\mathsf{V}^{\otimes k})$ is the planar rook algebra $\mathbb{C}\mathsf{P}_{k-1}$ for all $k \geq 1$, and we exhibit an explicit decomposition of $\mathsf{V}^{\otimes k}$ into irreducible $\mathfrak{gl}(1|1)$-modules. We obtain similar results for the quantum enveloping algebra $\mathsf{U}_{\mathsf{q}}(\mathfrak{gl}(1|1))$ and its natural two-dimensional module V_{q}.

Keywords: planar rook algebra, general linear Lie superalgebra, $\mathfrak{gl}(1|1)$-representation, quantum enveloping algebra $\mathsf{U}_{\mathsf{q}}(\mathfrak{gl}(1|1))$

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1 Introduction

The general linear Lie algebra $\mathfrak{gl}(m)$ of $m \times m$ matrices over the complex numbers $\mathbb{C}$ has a natural action on the space V of $m \times 1$ matrices by matrix multiplication. The symmetric group S_k acts by place permutations on the tensor power $\mathsf{V}^{\otimes k}$ and commutes with the action of $\mathfrak{gl}(m)$. Schur [S1, S2] exploited this commuting action to accomplish the decomposition of $\mathsf{V}^{\otimes k}$ into irreducible summands, and by doing this for all k, constructed all the irreducible polynomial representations of $\mathfrak{gl}(m)$. The commuting action determines an algebra epimorphism $\phi : \mathbb{C}\mathsf{S}_k \to \mathsf{End}_{\mathfrak{gl}(m)}(\mathsf{V}^{\otimes k})$, which is an isomorphism whenever $m \geq k$. In the special case when $m = 2$, the centralizer algebra $\mathsf{End}_{\mathfrak{gl}(2)}(\mathsf{V}^{\otimes k})$ is a quotient of the group algebra $\mathbb{C}\mathsf{S}_k$, which is the Temperley-Lieb algebra $\mathsf{TL}_k(2)$.

In their paper [TL] on statistical mechanical models in physics, H.N.V. Temperley and E.H. Lieb introduced what are now known as the Temperley-Lieb algebras. These algebras play a prominent role in Jones's work ([J1], [J3]) on subfactors of von Neumann algebras. The Temperley-Lieb algebra $\mathsf{TL}_k(\mathsf{q}+\mathsf{q}^{-1})$ also can be viewed as a quotient of the Iwahori-Hecke algebra $\mathcal{H}_k(\mathsf{q})$ of type A (the above is the special case corresponding to the parameter $\mathsf{q} = 1$), and that realization led Jones [J2] to discover an invariant of knots and links (the Jones polynomial). Kauffman's bracket model for the Jones polynomial has a tangle theoretic interpretation using Temperley-Lieb algebras, and this is closely connected with

the recoupling theory of colored knots and links and topological invariants of 3-manifolds [KL].

By comparison, the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ over $\mathbb{C}$ has a natural action on the space V of $(m+n) \times 1$ matrices by matrix multiplication. The symmetric group S_k acts by *graded* place permutations on the tensor power $\mathsf{V}^{\otimes k}$ and commutes with the $\mathfrak{gl}(m|n)$-action. As Berele and Regev show in [BR], the S_k-action yields an algebra epimorphism $\phi : \mathbb{C}\mathsf{S}_k \to \mathsf{End}_{\mathfrak{gl}(m|n)}(\mathsf{V}^{\otimes k})$, which is an isomorphism whenever $(m+1)(n+1) > k$. This can be used to decompose $\mathsf{V}^{\otimes k}$ into irreducible summands (see [BL]), and it enabled Berele and Regev to relate the characters of the irreducible $\mathfrak{gl}(m|n)$-summands to hook Schur functions.

In this work, we examine the case $m = 1 = n$, which is analogous to the $\mathfrak{gl}(2)$-case. We show that the planar rook algebra $\mathbb{C}\mathsf{P}_{k-1}$ is the centralizer algebra $\mathsf{End}_{\mathfrak{gl}(1|1)}(\mathsf{V}^{\otimes k})$ and use that insight to give an explicit decomposition of $\mathsf{V}^{\otimes k}$ into irreducible summands. The planar rook algebra, like its Temperley-Lieb counterpart, has a basis of diagrams. It has a rich combinatorics and irreducible modules whose dimensions are given by binomial coefficients (see [FHH]). The purpose of this short note is to tie planar rook algebras to the representation theory of the general linear Lie superalgebra $\mathfrak{gl}(1|1)$ and its quantum analogue. In Section 6 of the paper, we consider the quantum enveloping algebra $\mathsf{U_q}(\mathfrak{gl}(1|1))$ for q not a root of unity and tensor powers of its natural two-dimensional representation $\mathsf{V_q}$. Our main results in that section are an explicit decomposition of $\mathsf{V}_\mathsf{q}^{\otimes k}$ into irreducible modules for $\mathsf{U_q}(\mathfrak{gl}(1|1))$ and a proof that $\mathsf{End}_{\mathsf{U_q}(\mathfrak{gl}(1|1))}(\mathsf{V}_\mathsf{q}^{\otimes k})$ is also isomorphic to $\mathbb{C}\mathsf{P}_{k-1}$.

In the quantum case, there is an algebra homomorphism from the Iwahori-Hecke algebra $\mathcal{H}_k(\mathsf{q}^2)$ of type A onto the centralizer algebra $\mathsf{End}_{\mathsf{U_q}(\mathfrak{gl}(m|n))}(\mathsf{V}_\mathsf{q}^{\otimes k})$ for any m and n, where here $\mathsf{V_q}$ is the natural $(m+n)$-dimensional representation of $\mathsf{U_q}(\mathfrak{gl}(m|n))$. The irreducible $\mathsf{U_q}(\mathfrak{gl}(m|n))$-summands of $\mathsf{V}_\mathsf{q}^{\otimes k}$ have highest weights labeled by $(m|n)$-hook shape partitions of k, and the corresponding highest weight vectors were constructed in [M] using Gyoja's q-analogue of Young symmetrizers in $\mathcal{H}_k(\mathsf{q}^2)$.

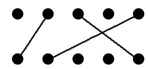
In the special case when $m = 1, n = 1$, Black, in recent work [B], applied Young's semi-normal representation of $\mathcal{H}_k(\mathsf{q}^2)$ on tableaux of hook shape (i.e. (1|1)-hook shape) to obtain an action of $\mathcal{H}_k(\mathsf{q}^2)$ on sequences of length k with components $+$ or $-$. The matrix units relative to a basis indexed by such sequences are described by permutation diagrams with strands colored by $+$ or $-$. Using the fact that there is a homomorphism $\mathsf{B}_k \to \mathcal{H}_k(\mathsf{q}^2)$ from the braid group B_k on k strands and identifying an element of B_k with a link, Black calculates a state-sum formula for the Alexander polynomial by taking traces of the action on sign sequences.

Our aim here is to determine the centralizer algebras $\mathsf{End}_{\mathfrak{gl}(1|1)}(\mathsf{V}^{\otimes k})$ and $\mathsf{End}_{\mathsf{U_q}(\mathfrak{gl}(1|1))}(\mathsf{V}_\mathsf{q}^{\otimes k})$ explicitly by realizing them as planar rook algebras and to provide an explicit decomposition of tensor space in both cases; problems not considered in [B]. In the final section of the paper we will explain how the colored permutation diagrams in [B] are related to the planar rook diagrams in our approach.

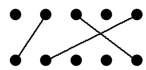
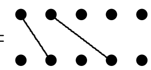
2 Planar Rook Algebras

The rook monoid R_n consists of all $n \times n$ matrices with at most one 1 in each row and column and 0 everywhere else. Each matrix in R_n corresponds to an arrangement of nonattacking rooks on an $n \times n$ chessboard. The permutation matrices inside R_n give a copy of the symmetric group S_n. Each matrix in R_n corresponds to an n-diagram; that is, a diagram

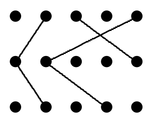
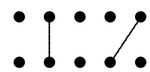
with two rows with n vertices in each and with edges that reflect the positions of the 1s:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow$$  .

Matrix multiplication in R_n corresponds to concatenation of diagrams; so for example, if

$d_1 =$  and $d_2 =$  ,

then

$d_1 d_2 \quad =$  $=$  .

The rook algebra $\mathbb{C}\mathsf{R}_n$ has as a basis the elements of R_n, and its unit element is just the diagram with n vertical edges, which corresponds to the identity matrix. As shown in ([Gr], [H], [HL], [So]), the algebra $\mathbb{C}\mathsf{R}_n$ has many interesting combinatorial features and representation theoretic connections which come from the fact $\mathbb{C}\mathsf{R}_n$ generates the centralizer algebra $\mathsf{End}_{\mathfrak{gl}(m)}((\mathbb{C}^m \oplus \mathbb{C})^{\otimes n})$ for all $m \geq 1$.

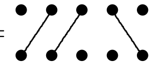
The n-diagrams with no edge crossings form a submonoid P_n of R_n, called the *planar rook monoid*. This can be regarded as the monoid of all order-preserving partial permutations of $\{1, 2, \ldots, n\}$, see [R]. The *planar rook algebra* $\mathbb{C}\mathsf{P}_n$ has as a basis over $\mathbb{C}$ the diagrams in P_n and multiplication that is the linear extension of the product in P_n. By convention, $\mathbb{C}\mathsf{P}_0 = \mathbb{C}$.

To count the number of diagrams in P_n with exactly ℓ edges, we choose ℓ vertices from each row of the diagram, and then connect those vertices in the one and only one planar way (the first on the bottom to the first on the top, the second to the second, etc.). There are exactly $\binom{n}{\ell}^2$ such diagrams, and from this we see

$$\dim \mathbb{C}\mathsf{P}_n = |\mathsf{P}_n| = \sum_{\ell=0}^{n} \binom{n}{\ell}^2 = \sum_{\ell=0}^{n} \binom{n}{\ell}\binom{n}{n-\ell} = \binom{2n}{n}.$$

The last equality is a special case of the Chu-Vandermonde identity and can be easily deduced from examining the coefficient of z^n in the expansion of $(1+z)^n(1+z)^n = (1+z)^{2n}$.

Given a diagram d in P_n, let $\beta(d)$ (resp. $\tau(d)$) be the vertices in the bottom (resp. top) row with edges emanating from them. Then d can be regarded as an order-preserving bijective map $d : \beta(d) \to \tau(d)$ with domain $\beta(d)$ and image $\tau(d)$. Thus, for the diagram

$d =$  ,

$\beta(d) = \{1, 2, 5\}$ and $\tau(d) = \{2, 3, 4\}$.

When s and t are subsets of $\{1, \ldots, n\}$ such that $|\mathsf{s}| = |\mathsf{t}|$, we adopt the notation $d^{\mathsf{s}}_{\mathsf{t}}$ for the diagram with $\beta(d^{\mathsf{s}}_{\mathsf{t}}) = \mathsf{t}$ and $\tau(d^{\mathsf{s}}_{\mathsf{t}}) = \mathsf{s}$.

Following [FHH], we construct a $\mathbb{C}\mathsf{P}_n$-module $\mathsf{M} = \bigoplus_{\mathsf{s}\subseteq\{1,\dots,n\}} \mathbb{C}m_{\mathsf{s}}$ with basis indexed by the subsets s of $\{1,\dots,n\}$, where the $\mathbb{C}\mathsf{P}_n$-action is defined by

$$dm_{\mathsf{s}} = \begin{cases} m_{d\mathsf{s}} & \text{if } \ \mathsf{s} \subseteq \beta(d) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $dm_{\emptyset} = m_{\emptyset}$. In particular, for the diagram d pictured above, $dm_{\{2,5\}} = m_{\{3,4\}}$, while $dm_{\{4,5\}} = 0$.

Proposition 2.1. ([FHH]) *Let $\mathsf{M}_\ell = \bigoplus_{|\mathsf{s}|=\ell} \mathbb{C}m_{\mathsf{s}} \subseteq \mathsf{M}$ for $\ell = 0,1,\dots,n$. Then M_ℓ is an irreducible $\mathbb{C}\mathsf{P}_n$-submodule of M of dimension $\binom{n}{\ell}$ and $\mathsf{M} = \bigoplus_{\ell=0}^{n} \mathsf{M}_\ell$.*

Proof. Assume N is a nonzero submodule of M_ℓ and $0 \neq m' = \sum_{\mathsf{s},|\mathsf{s}|=\ell} \xi_{\mathsf{s}} m_{\mathsf{s}} \in \mathsf{N}$. We suppose $\xi_{\mathsf{t}} \neq 0$. Then $d_{\mathsf{t}}^{\mathsf{t}} m' = \sum_{\mathsf{s},|\mathsf{s}|=\ell} \xi_{\mathsf{s}} d_{\mathsf{t}}^{\mathsf{t}} m_{\mathsf{s}} = \xi_{\mathsf{t}} m_{\mathsf{t}} \in \mathsf{N}$, because in order for $d_{\mathsf{t}}^{\mathsf{t}} m_{\mathsf{s}}$ to be nonzero, $\mathsf{s} \subseteq \beta(d_{\mathsf{t}}^{\mathsf{t}}) = \mathsf{t}$ must hold; but $|\mathsf{s}| = |\mathsf{t}|$, so $\mathsf{s} = \mathsf{t}$ is forced. Hence, $m_{\mathsf{t}} \in \mathsf{N}$ and so is $d_{\mathsf{t}}^{\mathsf{s}} m_{\mathsf{t}} = m_{\mathsf{s}} \in \mathsf{N}$ for all s with $|\mathsf{s}| = \ell$. From this it follows that any nonzero submodule N of M_ℓ must equal M_ℓ, so M_ℓ is irreducible. The rest of the assertions follow readily. $\square$

Proposition 2.2. ([FHH, Thm. 3.2]) *The modules M_ℓ for $\ell = 0,1\dots,n$ form a complete set of irreducible $\mathbb{C}\mathsf{P}_n$-modules. As a $\mathbb{C}\mathsf{P}_n$-module, $\mathbb{C}\mathsf{P}_n \cong \bigoplus_{\ell=0}^{n} \binom{n}{\ell} \mathsf{M}_\ell$. Thus, the regular representation of $\mathbb{C}\mathsf{P}_n$ is completely reducible, and $\mathbb{C}\mathsf{P}_n$ is a semisimple associative algebra.*

3 $\mathfrak{gl}(1|1)$

The general linear Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(1|1) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ over $\mathbb{C}$ consists of all 2×2 complex matrices under the commutator product $[a,b] = ab - (-1)^{\alpha\beta} ba$ for $a \in \mathfrak{g}_\alpha, b \in \mathfrak{g}_\beta$ and $\alpha,\beta \in \mathbb{Z}/2\mathbb{Z}$. This product is bilinear and satisfies $[b,a] = -(-1)^{\alpha\beta}[a,b]$ and the super-Jacobi identity $[a,[b,c]] = [[a,b],c] + (-1)^{\alpha\beta}[b,[a,c]]$ for all $a \in \mathfrak{g}_\alpha, b \in \mathfrak{g}_\beta, c \in \mathfrak{g}$.

The elements

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

constitute a basis for $\mathfrak{gl}(1|1)$, and $\mathfrak{g}_0 = \mathbb{C}h_1 \oplus \mathbb{C}h_2$ and $\mathfrak{g}_1 = \mathbb{C}e \oplus \mathbb{C}f$. It is easy to check that the following relations hold:

$$[e,f] = ef + fe = h_1 + h_2 = I, \quad [h_1,e] = e, \quad [h_2,e] = -e, \quad [h_1,f] = -f, \quad [h_2,f] = f.$$

In any Lie superalgebra, an odd degree element a satisfies $a^2 = \frac{1}{2}[a,a]$ in the universal enveloping algebra. In particular, in the universal enveloping algebra of $\mathfrak{gl}(1|1)$, we have

$$e^2 = \frac{1}{2}[e,e] = 0 = \frac{1}{2}[f,f] = f^2, \tag{3.1}$$

because computing the commutators $[e,e]$ and $[f,f]$ in $\mathfrak{gl}(1|1)$ involves squaring off-diagonal matrix units, which gives 0.

Each finite-dimensional irreducible $\mathfrak{gl}(1|1)$-module has a unique (up to scalar multiple) highest weight vector; that is, a nonzero vector w such that $ew = 0$ and $h_1 w = mw$, $h_2 w = nw$ for some integers m,n. The module is uniquely determined by the pair $[m,n]$, and we denote it by $\mathsf{L}[m,n]$.

Here we consider the two-dimensional irreducible $\mathfrak{gl}(1|1)$-module $\mathsf{V} = \mathsf{V}_0 \oplus \mathsf{V}_1$, where $\mathsf{V}_0 = \mathbb{C}x$ and $\mathsf{V}_1 = \mathbb{C}y$, and its k-fold tensor power $\mathsf{V}^{\otimes k}$ for all $k \geq 1$. We can identify V with the 2×1 matrices over $\mathbb{C}$, and x and y with the standard unit vectors

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where the action of $\mathfrak{gl}(1|1)$ is given by matrix multiplication:

$$\begin{aligned} ex &= 0, \quad ey = x, \qquad fx = y, \quad fy = 0, \\ h_1 x &= x, \quad h_1 y = 0, \qquad h_2 x = 0, \quad h_2 y = y. \end{aligned}$$

In the notation above, $\mathsf{V} = \mathsf{L}[1,0]$ and x is the highest weight vector of weight $[1,0]$.

Now it is not difficult to argue that

$$\mathsf{L}[m,n] \otimes \mathsf{V} = \mathsf{L}[m+1,n] \oplus \mathsf{L}[m,n+1].$$

Starting with V and iterating, we see that

$$\begin{aligned} \mathsf{V}^{\otimes 2} &= \mathsf{L}[2,0] \oplus \mathsf{L}[1,1] \\ \mathsf{V}^{\otimes 3} &= \mathsf{L}[3,0] \oplus 2\mathsf{L}[2,1] \oplus \mathsf{L}[1,2] \\ \mathsf{V}^{\otimes 4} &= \mathsf{L}[4,0] \oplus 3\mathsf{L}[3,1] \oplus 3\mathsf{L}[2,2] \oplus \mathsf{L}[1,3] \\ &\vdots \\ \mathsf{V}^{\otimes k} &= \bigoplus_{\ell=0}^{k-1} \binom{k-1}{\ell} \mathsf{L}[\ell+1, k-1-\ell] \end{aligned}$$

The last line follows from an easy inductive argument which shows that the multiplicity of $\mathsf{L}[\ell+1, k-1-\ell]$ in $\mathsf{V}^{\otimes k}$ is equal to the number of paths of length $k-1$ from $\mathsf{L}[1,0] = \mathsf{V}$ to $\mathsf{L}[\ell+1, k-1-\ell]$ with ℓ steps in which 1 is added to the first component of the highest weight and $k-1-\ell$ steps with 1 added to the second component.

Since the $\mathfrak{gl}(1|1)$-module $\mathsf{V}^{\otimes k}$ is completely reducible, by the classical double-centralizer theory (see for example [CR, Secs. 3B and 68]) the centralizer algebra $\mathsf{C}_k = \mathsf{End}_{\mathfrak{gl}(1|1)}(\mathsf{V}^{\otimes k})$ is a semisimple associative algebra over $\mathbb{C}$, hence the sum of matrix blocks of size $\binom{k-1}{\ell} \times \binom{k-1}{\ell}$, and

$$\dim \mathsf{C}_k = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell}^2 = \binom{2(k-1)}{k-1}.$$

Thus, C_k is isomorphic to the planar rook algebra $\mathbb{C}\mathsf{P}_{k-1}$. In what follows, we want to exhibit the decomposition of $\mathsf{V}^{\otimes k}$ into irreducible $\mathfrak{gl}(1|1)$-summands and to demonstrate an explicit action of the planar rook algebra $\mathbb{C}\mathsf{P}_{k-1}$ on $\mathsf{V}^{\otimes k}$ which commutes with the $\mathfrak{gl}(1|1)$-action.

4 The Decomposition of $\mathsf{V}^{\otimes k}$

Our goal here is to display the decomposition of $\mathsf{V}^{\otimes k}$ for $\mathsf{V} = \mathbb{C}x \oplus \mathbb{C}y$ into irreducible $\mathfrak{gl}(1|1)$-submodules explicitly. We use the fact that the action on a tensor product is given

by

$$a(v \otimes w) = av \otimes w + (-1)^{\alpha\beta} v \otimes aw$$

for $a \in \mathfrak{gl}(1|1)$ of degree α and $v \in \mathsf{V}$ of degree β.

Assume s is a subset of $\{1, \dots, k-1\}$ and define:

$$\begin{aligned} u_{\mathsf{s}} &= u_1 \otimes \cdots \otimes u_{k-1} && (4.1) \\ v_{\mathsf{s}} &= e(y \otimes u_{\mathsf{s}}) = x \otimes u_{\mathsf{s}} - y \otimes eu_{\mathsf{s}}, && (4.2) \end{aligned}$$

where for $i = 1, \dots, k-1$,

$$u_i = \begin{cases} x & \text{if } i \in \mathsf{s} \\ y & \text{if } i \notin \mathsf{s}. \end{cases}$$

For example, when $k = 4$ and $\mathsf{s} = \{1, 3\}$, we have $u_{\mathsf{s}} = x \otimes y \otimes x$ and $v_{\mathsf{s}} = x \otimes x \otimes y \otimes x - y \otimes x \otimes x \otimes x$.

Claim 1. *For each subset $\mathsf{s} \subseteq \{1, \dots, k-1\}$, the vectors v_{s} and fv_{s} span a two-dimensional irreducible $\mathfrak{gl}(1|1)$-submodule V_{s} of $\mathsf{V}^{\otimes k}$ with highest weight vector v_{s} of weight $[|\mathsf{s}| + 1, k - 1 - |\mathsf{s}|]$.*

Proof. Observe first that $v_{\mathsf{s}} \neq 0$ and $ev_{\mathsf{s}} = e^2(y \otimes u_{\mathsf{s}}) = 0$ by (3.1). Similarly, $f(fv_{\mathsf{s}}) = f^2 v_{\mathsf{s}} = 0$. Next, we have

$$e(fv_{\mathsf{s}}) = efv_{\mathsf{s}} = -fev_{\mathsf{s}} + Iv_{\mathsf{s}} = 0 + kv_{\mathsf{s}} \neq 0,$$

so in particular, $fv_{\mathsf{s}} \neq 0$. Now

$$h_1 v_{\mathsf{s}} = h_1 e(y \otimes u_{\mathsf{s}}) = eh_1(y \otimes u_{\mathsf{s}}) + e(y \otimes u_{\mathsf{s}}) = (|\mathsf{s}| + 1)v_{\mathsf{s}}$$

and

$$h_2 v_{\mathsf{s}} = h_2 e(y \otimes u_{\mathsf{s}}) = eh_2(y \otimes u_{\mathsf{s}}) - e(y \otimes u_{\mathsf{s}}) = (k - 1 - |\mathsf{s}|)v_{\mathsf{s}}.$$

Similarly, $h_1 fv_{\mathsf{s}} = |\mathsf{s}| fv_{\mathsf{s}}$ and $h_2 fv_{\mathsf{s}} = (k - |\mathsf{s}|)fv_{\mathsf{s}}$.

It follows that V_{s} is a two-dimensional submodule of $\mathsf{V}^{\otimes k}$; v_{s} is a highest weight vector of weight $[|\mathsf{s}| + 1, k - 1 - |\mathsf{s}|]$; and fv_{s} is a lowest weight vector of weight $[||\mathsf{s}|, k - |\mathsf{s}|]$. It is easy to argue that V_{s} is irreducible, since any nonzero submodule must decompose into weight spaces relative to h_1 and h_2, and so must contain either v_{s} or fv_{s}, hence both of them. Therefore, all the assertions in Claim 1 hold.

Claim 2. $\mathsf{V}^{\otimes k} = \bigoplus_{\mathsf{s}} \mathsf{V}_{\mathsf{s}}$*, where s ranges over the subsets of $\{1, \dots, k-1\}$, affords a decomposition of $\mathsf{V}^{\otimes k}$ into irreducible $\mathfrak{gl}(1|1)$-modules.*

Proof. If $0 \neq \mathsf{V}_{\mathsf{t}} \cap \sum_{\mathsf{s} \neq \mathsf{t}} \mathsf{V}_{\mathsf{s}}$ for some subset t, then V_{t} must be contained in the right-hand sum by its irreducibility. Hence, v_{t} must be a nonzero sum of highest weight vectors in the sum; that is, it must be a linear combination of the vectors v_{s} with $\mathsf{s} \neq \mathsf{t}$ (and in fact, with $|\mathsf{s}| = |\mathsf{t}|$). However, the terms $x \otimes u_{\mathsf{s}}$ in the vectors v_{s} in (4.2) are all linearly independent. This contradiction shows that the sum $\sum_{\mathsf{s}} \mathsf{V}_{\mathsf{s}}$ is direct. But $\dim\left(\bigoplus_{\mathsf{s}} \mathsf{V}_{\mathsf{s}}\right) = 2^{k-1} \times 2 = 2^k = \dim(\mathsf{V}^{\otimes k})$, so Claim 2 holds.

Theorem 4.3. $\displaystyle \mathsf{V}^{\otimes k} = \bigoplus_{\mathsf{s}} \mathsf{V}_{\mathsf{s}} = \bigoplus_{\ell=0}^{k-1} \binom{k-1}{\ell} \mathsf{L}[\ell + 1, k - 1 - \ell].$

Proof. This is an immediate consequence of Claims 1 and 2. □

5 The action of $\mathbb{C}\mathsf{P}_{k-1}$ on $\mathsf{V}^{\otimes k}$

It follows from the previous section that $\{v_{\mathsf{s}}, fv_{\mathsf{s}} \mid \mathsf{s} \subseteq \{1, \ldots, k-1\}\}$ is a basis for $\mathsf{V}^{\otimes k}$. Given a diagram $d \in \mathsf{P}_{k-1}$, we define an action of d on $\mathsf{V}^{\otimes k}$ by specifying its action on this basis according to

$$dv_{\mathsf{s}} = v_{d\mathsf{s}}, \qquad d(fv_{\mathsf{s}}) = fv_{d\mathsf{s}} = f(dv_{\mathsf{s}}), \tag{5.1}$$

and then extend the action linearly from P_{k-1} to all of $\mathbb{C}\mathsf{P}_{k-1}$. Since $d_1(d_2\mathsf{s}) = (d_1d_2)\mathsf{s}$ holds for all $d_1, d_2 \in \mathsf{P}_{k-1}$ and all subsets $\mathsf{s} \subseteq \{1, \ldots, k-1\}$, this makes $\mathsf{V}^{\otimes k}$ into a module for the algebra $\mathbb{C}\mathsf{P}_{k-1}$.

Claim 3. *The $\mathbb{C}\mathsf{P}_{k-1}$-action and the $\mathfrak{gl}(1|1)$-action on $\mathsf{V}^{\otimes k}$ commute.*

Proof. Observe that $h_1dv_{\mathsf{s}} = h_1v_{d\mathsf{s}} = |d\mathsf{s}|v_{d\mathsf{s}} = |\mathsf{s}|v_{d\mathsf{s}} = dh_1v_{\mathsf{s}}$, and similarly for h_2. As $dev_{\mathsf{s}} = 0 = ev_{d\mathsf{s}} = edv_{\mathsf{s}}$, it is apparent from this and (5.1) that d commutes with the action of $\mathfrak{gl}(1|1)$ on all the vectors v_{s}. It also commutes with the action of $\mathfrak{gl}(1|1)$ on the vectors fv_{s}. For example,

$$de(fv_{\mathsf{s}}) = d(efv_{\mathsf{s}}) = d(Iv_{\mathsf{s}}) = kdv_{\mathsf{s}} = kv_{d\mathsf{s}} = efv_{d\mathsf{s}} = ef(dv_{\mathsf{s}}) = ed(fv_{\mathsf{s}}).$$

We leave the rest of the verifications to the reader.

Claim 4. *If $p \in \mathbb{C}\mathsf{P}_{k-1}$ has the property that $pw = 0$ for all $w \in \mathsf{V}^{\otimes k}$, then $p = 0$.*

Proof. We may suppose that $p = \sum_{\mathsf{s},\mathsf{t}} \zeta_{\mathsf{t}}^{\mathsf{s}} d_{\mathsf{t}}^{\mathsf{s}}$ where $\zeta_{\mathsf{t}}^{\mathsf{s}} \in \mathbb{C}$ and s, t are subsets of $\{1, \ldots, k-1\}$ with $|\mathsf{s}| = |\mathsf{t}|$. We assume t' is chosen so $|\mathsf{t}'|$ is maximal among all subsets t with $\zeta_{\mathsf{t}}^{\mathsf{s}} \neq 0$ for some s. Then

$$0 = pv_{\mathsf{t}'} = \sum_{\mathsf{s}, |\mathsf{s}|=|\mathsf{t}'|} \zeta_{\mathsf{t}'}^{\mathsf{s}} d_{\mathsf{t}'}^{\mathsf{s}} v_{\mathsf{t}'} = \sum_{\mathsf{s}, |\mathsf{s}|=|\mathsf{t}'|} \zeta_{\mathsf{t}'}^{\mathsf{s}} v_{\mathsf{s}},$$

which by the independence of the vectors v_{s} forces $\zeta_{\mathsf{t}'}^{\mathsf{s}} = 0$, a contradiction. Thus, $p = 0$.

Theorem 5.2. $\mathsf{End}_{\mathfrak{gl}(1|1)}(\mathsf{V}^{\otimes k}) = \mathbb{C}\mathsf{P}_{k-1}$

Proof. Claims 3 and 4 imply that we may regard $\mathbb{C}\mathsf{P}_{k-1}$ as a subalgebra of $\mathsf{End}_{\mathfrak{gl}(1|1)}(\mathsf{V}^{\otimes k})$. However

$$\dim(\mathbb{C}\mathsf{P}_{k-1}) = \binom{2(k-1)}{k-1} = \dim(\mathsf{End}_{\mathfrak{gl}(1|1)}(\mathsf{V}^{\otimes k})),$$

so that equality is forced. □

6 Tensor representations for quantum $\mathfrak{gl}(1|1)$

In this section we introduce the quantum enveloping algebra $\mathsf{U_q}(\mathfrak{gl}(1|1))$ and study tensor powers of its natural two-dimensional module $\mathsf{V_q}$. When q is not a root of unity $\mathsf{V}_\mathsf{q}^{\otimes k}$ is completely reducible, and we display its decomposition into irreducible summands. The centralizer algebra of the $\mathsf{U_q}(\mathfrak{gl}(1|1))$-action on $\mathsf{V}_\mathsf{q}^{\otimes k}$ is shown to be the planar rook algebra $\mathbb{C}\mathsf{P}_{k-1}$ just as in the $\mathfrak{gl}(1|1)$ case.

Let P be the free $\mathbb{Z}$-module given by $\mathsf{P} = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$ and let $\mathsf{P}^* = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2$ be the dual module under the natural bilinear pairing $\langle\ ,\ \rangle$ for which $\langle h_i, \varepsilon_j\rangle = \delta_{i,j}$. We set $\alpha = \varepsilon_1 - \varepsilon_2$, and let q be a nonzero element of $\mathbb{C}$. We consider the unital Hopf algebra $\mathsf{U_q} = \mathsf{U_q}(\mathfrak{gl}(1|1))$ over $\mathbb{C}$ with generators $E, F, \sigma, q^h (h \in \mathsf{P}^*)$ and relations

$$\begin{gathered}
q^h = 1 \quad (\text{for} \quad h = 0), \qquad q^h q^{h'} = q^{h+h'}, \\
q^h E = \mathsf{q}^{\langle h,\alpha\rangle} E q^h, \qquad q^h F = \mathsf{q}^{-\langle h,\alpha\rangle} F q^h, \\
EF + FE = \frac{K - K^{-1}}{\mathsf{q} - \mathsf{q}^{-1}} \quad \text{where} \quad K = q^{h_1+h_2}, \\
E^2 = 0 = F^2, \\
\sigma E = -E\sigma, \quad \sigma F = -F\sigma, \quad \sigma q^h = q^h \sigma, \quad \sigma^2 = 1,
\end{gathered}$$

for all $h, h' \in \mathsf{P}^*$. The coproduct Δ, counit u, and antipode S on $\mathsf{U_q}$ are given by

$$\begin{gathered}
\Delta(E) = E \otimes K^{-1} + \sigma \otimes E, \quad \Delta(F) = F \otimes 1 + \sigma K \otimes F \\
\Delta(q^h) = q^h \otimes q^h, \qquad \Delta(\sigma) = \sigma \otimes \sigma \\
\mathsf{u}(E) = 0 = \mathsf{u}(F) = 0, \quad \mathsf{u}(K) = 1 = \mathsf{u}(\sigma) \\
S(E) = -\sigma E K, \quad S(F) = -\sigma K^{-1} F, \quad S(q^h) = q^{-h}, \quad S(\sigma) = \sigma.
\end{gathered}$$

The algebra $\mathsf{U_q}$ is the quantum enveloping algebra of the Lie superalgebra $\mathfrak{gl}(1|1)$ studied in [BKK].

Here we consider the two-dimensional module $\mathsf{V_q} = \mathbb{C}x \oplus \mathbb{C}y$ for $\mathsf{U_q}$ and the k-fold tensor power $\mathsf{V}_\mathsf{q}^{\otimes k}$ of $\mathsf{V_q}$. The action of $\mathsf{U_q}$ on $\mathsf{V_q}$ is given by

$$\begin{gathered}
Ex = 0, \quad Ey = x, \qquad Fx = y, \quad Fy = 0, \\
q^h x = \mathsf{q}^{\langle h,\varepsilon_1\rangle} x, \quad q^h y = \mathsf{q}^{\langle h,\varepsilon_2\rangle} y, \qquad \sigma(x) = x, \quad \sigma(y) = -y.
\end{gathered}$$

The coproduct gives the action of $\mathsf{U_q}$ on a tensor product of any two $\mathsf{U_q}$-modules.

Imitating the $\mathfrak{gl}(1|1)$ case, we consider subsets s of $\{1, \ldots, k-1\}$ and define

$$u_\mathsf{s} = u_1 \otimes \cdots \otimes u_{k-1} \tag{6.1}$$

$$v_\mathsf{s} = E(y \otimes u_\mathsf{s}) = x \otimes K^{-1} u_\mathsf{s} - y \otimes E u_\mathsf{s} = \mathsf{q}^{1-k} x \otimes u_\mathsf{s} - y \otimes E u_\mathsf{s}, \tag{6.2}$$

where for $i = 1, \ldots, k-1$,

$$u_i = \begin{cases} x & \text{if } i \in \mathsf{s} \\ y & \text{if } i \notin \mathsf{s}. \end{cases}$$

For example, when $k = 3$ and $\mathsf{s} = \{1\}$, we have $u_{\mathsf{s}} = x \otimes y$ and

$$v_{\mathsf{s}} = E(y \otimes x \otimes y) = \mathsf{q}^{-2} x \otimes x \otimes y - y \otimes x \otimes x.$$

Claim 5. *For each subset $\mathsf{s} \subseteq \{1, \ldots, k-1\}$, the vectors v_{s} and Fv_{s} span a two-dimensional irreducible $\mathsf{U_q}$-submodule $\mathsf{V_{q,s}}$ of $\mathsf{V}_{\mathsf{q}}^{\otimes k}$ with highest weight vector v_{s} of highest weight $[\mathsf{q}^{|s|+1}, \mathsf{q}^{k-1-|s|}]$ relative to q^{h_1}, q^{h_2} whenever q is not a root of unity.*

Proof. First note that $Ev_{\mathsf{s}} = E^2(y \otimes u_{\mathsf{s}}) = 0$, while

$$q^{h_1} v_{\mathsf{s}} = \mathsf{q}^{|\mathsf{s}|+1} v_{\mathsf{s}}, \quad q^{h_2} v_{\mathsf{s}} = \mathsf{q}^{k-1-|\mathsf{s}|} v_{\mathsf{s}}, \quad \sigma v_{\mathsf{s}} = (-1)^{k-|s|} v_{\mathsf{s}}.$$

Similarly $F(Fv_{\mathsf{s}}) = F^2 v_{\mathsf{s}} = 0$,

$$q^{h_1} F v_{\mathsf{s}} = \mathsf{q}^{|s|} v_{\mathsf{s}}, \quad q^{h_2} F v_{\mathsf{s}} = \mathsf{q}^{k-|s|} v_{\mathsf{s}}, \quad \sigma F v_{\mathsf{s}} = (-1)^{k+1-|s|} F v_{\mathsf{s}}.$$

Now

$$EFv_{\mathsf{s}} = -FEv_{\mathsf{s}} + \left(\frac{K - K^{-1}}{\mathsf{q} - \mathsf{q}^{-1}} \right) v_{\mathsf{s}} = [k] v_{\mathsf{s}}, \tag{6.3}$$

where $[k]$ is the q-integer given by

$$[k] := \frac{\mathsf{q}^k - \mathsf{q}^{-k}}{\mathsf{q} - \mathsf{q}^{-1}}.$$

From this it is apparent that $\mathsf{V_{q,s}}$ is a two-dimensional $\mathsf{U_q}$-module. It is irreducible, because any submodule must contain an eigenvector for σ hence either v_{s} or Fv_{s}. But since $[k] \neq 0$, it will then contain both vectors (see (6.3)).

Let $\mathsf{L_q}[m, n]$ denote the finite-dimensional irreducible $\mathsf{U_q}$-module with highest weight $[\mathsf{q}^m, \mathsf{q}^n]$ relative to h_1, h_2. Then $\mathsf{V_{q,s}} \cong \mathsf{L_q}[|s| + 1, k - 1 - |s|]$, and as in the $\mathfrak{gl}(1|1)$ case we have

Theorem 6.4. $\mathsf{V}_{\mathsf{q}}^{\otimes k} = \bigoplus_{\mathsf{s}} \mathsf{V_{q,s}} = \bigoplus_{\ell=0}^{k-1} \binom{k-1}{\ell} \mathsf{L_q}[\ell + 1, k - 1 - \ell]$, *where s ranges over the subsets of $\{1, \ldots, k-1\}$, affords a decomposition of $\mathsf{V}_{\mathsf{q}}^{\otimes k}$ into irreducible modules for $\mathsf{U_q} = \mathsf{U_q}(\mathfrak{gl}(1|1))$ whenever q is not a root of unity.*

Just as in the $\mathfrak{gl}(1|1)$ case, we may define an action of the planar rook algebra $\mathbb{C}\mathsf{P}_{k-1}$on $\mathsf{V}_{\mathsf{q}}^k$ by assigning for each diagram $d \in \mathsf{P}_{k-1}$,

$$dv_{\mathsf{s}} = v_{d\mathsf{s}} \qquad dFv_{\mathsf{s}} = Fv_{d\mathsf{s}} = Fdv_{\mathsf{s}}.$$

Theorem 6.5. *When q is not a root of unity, we have $\mathsf{End}_{\mathsf{U}_q}(\mathsf{V}_{\mathsf{q}}^{\otimes k}) = \mathbb{C}\mathsf{P}_{k-1}$ for $\mathsf{U_q} = \mathsf{U_q}(\mathfrak{gl}(1|1))$.*

Proof. The arguments to show that $\mathbb{C}\mathsf{P}_{k-1}$ commutes with the action of $\mathsf{U_q}$ on $\mathsf{V_q}$ and to show that $\mathbb{C}\mathsf{P}_{k-1}$ embeds into the centralizer algebra $\mathsf{End}_{\mathsf{U}_q}(\mathsf{V}_{\mathsf{q}}^{\otimes k})$ are virtually identical to the $\mathfrak{gl}(1|1)$ case. Here is one sample computation to show that a diagram $d \in \mathsf{P}_{k-1}$ commutes with E on a vector Fv_{s}:

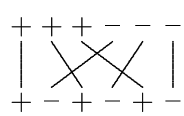
$$
\begin{aligned}
(dE)Fv_{\mathsf{s}} &= d(EF)v_{\mathsf{s}} = -d(FE)v_{\mathsf{s}} + d\left(\frac{K-K^{-1}}{\mathsf{q}-\mathsf{q}^{-1}}\right)v_{\mathsf{s}} \\
&= [k]dv_{\mathsf{s}} = [k]v_{d\mathsf{s}} = \left(\frac{K-K^{-1}}{\mathsf{q}-\mathsf{q}^{-1}}\right)v_{d\mathsf{s}} \\
&= EFv_{d\mathsf{s}} = EFdv_{\mathsf{s}} = EdFv_{\mathsf{s}} \\
&= (Ed)Fv_{\mathsf{s}}.
\end{aligned}
$$

The remaining verifications are left to the reader. □

7 Connections with representations of $\mathcal{H}_k(\mathsf{q}^2)$

It is known (see [M], for example) that there is an algebra epimorphism $\varphi : \mathcal{H}_k(\mathsf{q}^2) \to \mathsf{End}_{\mathsf{U}_{\mathsf{q}}(\mathfrak{gl}(1|1))}(\mathsf{V}_{\mathsf{q}}^{\otimes k})$ giving a representation of the Iwahori-Hecke algebra $\mathcal{H}_k(\mathsf{q}^2)$ on $\mathsf{V}_{\mathsf{q}}^{\otimes k}$. The quotient $\mathcal{H}_k(\mathsf{q}^2)/\ker\varphi$ is semisimple and decomposes into matrix blocks indexed by the partitions of k of hook shape. Black [B] described an action of this quotient algebra on the span of vectors x_η indexed by sequences $\eta = (\eta_1, \dots, \eta_k)$ with $\eta_j = \pm$ for $j = 1, \dots, k$ such that $\eta_1 = +$. The sequences η having ℓ coordinates equal to $-$ are in one-to-one correspondence with the standard tableaux whose shape is given by the partition $(k-\ell, 1^\ell)$, and the action of $\mathcal{H}_k(\mathsf{q}^2)$ on the vectors x_η indexed by those sequences is derived from Young's semi-normal representation on the corresponding standard tableaux.

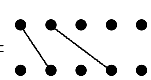
Now the matrix units relative to the basis x_η are in bijection with the permutation diagrams with strands colored $+$ and $-$ such that no two strands of the same color cross, and the first strand goes directly down and has color $+$. For example, the matrix unit $\mathsf{E}_{\vartheta,\eta}$ labeled by the sequences $\vartheta = (+,+,+,-,-,-)$ and $\eta = (+,-,+,-,+,-)$ corresponds to the permutation diagram



.

There is a homomorphism $\psi : \mathsf{B}_k \to \mathcal{H}_k(\mathsf{q}^2)$ from the braid group B_k on k-strands, and so the action above determines an action of B_k on the space spanned by the vectors x_η. Taking traces gives the Alexander polynomial of the link associated to a braid group element.

The correspondence between the colored permutation diagrams and the elements of the planar rook monoid P_{k-1} can be obtained by ignoring the first strand and deleting the strands colored $-$. The diagram above corresponds to the element $\mathsf{X}_d = \sum_{d' \subseteq d} (-1)^{|d \setminus d'|} d' \in \mathbb{C}\mathsf{P}_{k-1}$ in [FHH] for

$d =$  ,

where d' ranges over all the diagrams obtained from d by deleting edges, and $|d \setminus d'|$ is the number of edges in d minus the number of edges in d'. If d_1 and d_2 are two diagrams in P_{k-1}, then by [FHH, Prop. 3.3] we have

$$
\mathsf{X}_{d_1}\mathsf{X}_{d_2} = \delta_{\beta(d_1),\tau(d_2)}\mathsf{X}_{d_3},
$$

where d_3 is the diagram with top row $\tau(d_1)$ and bottom row $\beta(d_2)$. Therefore, the correspondence between the matrix units $\mathsf{E}_{\vartheta,\eta}$ and the matrix units X_d determines an algebra isomorphism between the algebra having basis the colored permutation diagrams on sequences of length k and the planar rook algebra $\mathbb{C}\mathsf{P}_{k-1}$.

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